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Topological Algebra via Inner Product

Azram, M.

Department of Science in Engineering,
International Islamic University Malaysia

E-mail: azram50@hotmail.com

ABSTRACT

This paper is devoted to establish a probability measure on a unital commutative separable Frechet Q lmc*- algebra. Consequently a new technique to define an inner product on a unital commutative semi simple separable Frechet Q lmc*-algebra. We have shown that the resulting inner product space is a topological algebra. At the end we have established some properties of the introduced inner product.

Keywords: Commutative semi simple Frechet algebra, Banach algebras, topological algebra, locally convex algebra.

1. Introduction

Banach Algebra is a very focused area of interest among researcher. Consequently, the subject has brought the theory to a point where it is no longer just a promising tool in its own right. Banach algebra has developed due to analytic and algebraic influences. The analytic emphasis has been on the study of some special Banach algebras, along with some generalization, and on extension of function theory and harmonic analysis to a more general situation. The algebraic emphasis has been on various aspects of structure theory. Several mathematicians have contributed to this subject and there is an impressive literature concerning the theory of Banach algebra and C*-algebras. It is natural to generalize this concept and discuss the validity of the theory of C*-algebras in a more general framework. The concept of topological algebra and hence a locally m-convex algebra is a natural generalization of Banach algebra. Dantzig has used the word "Topological Algebra' very first time in his Ph.D. thesis and later in a whole series of papers. Later on it also appeared in the literature by Arens (1947). Fundamental results, although at the same time but separately, were published by Arens (1952) and Michael (1952).The study of these algebras helped to investigate the non-normable behaviors in mathematics and physics. Michael (1952) gave the first systematic treatment of the theory in 1952. Since then, numerous papers have been written on the subject. Most of Michael's efforts were devoted to formulate and prove certain generalizations and well known fact about C*-algebra. However, it should be made clear that, although most of the generalizations are easily anticipated, many proofs require novel insights and substantial re-working to recover C*-algebra counterpart. Brooks (1967), Johnson (1982), Husain (1989), Ansari (2010) Azhari (2013) and many others discuss the continuity of positive functional on Topological*-algebras. They also have developed the details for the proof of Gelfand-Naimark-Segal (GNS) construction which plays an important role in the representation theory of locally m-convex* -algebras. There are several ways of defining an inner product on lmc*-algebra Brooks (1967), Fragoulopoulou (1981), Fragoulopoulou (2005) and Noor and Alberto. Analogue to Noor and Alberto, we will be introducing a new inner product on a locally multiplicatively *-algebra and consequently to discuss the new structure. This paper is devoted to establish a probability measure on a unital commutative separable Frechet Q lmc*- algebra. Consequently a new technique to define an inner product on a unital commutative semi simple separable Frechet Q lmc*-algebra. We have shown that the resulting inner product space is a topological algebra. At the end we have established some properties of the introduced inner product. I suspect this will reduce the theory of locally multiplicatively *-algebra to the theory of Hilbert space, because of their notably application in analysis and also in differential geometry, Non-commutative Geometry Connes (1994) group theory, stochastic, theoretical physics, quantum mechanics and even number theory.

2. Material and Methods

A vector space A over the complex field C is called complex algebra if

i. A is closed i.e., $\forall x, y \in A$ xy is defined and $\in A$

ii. $x(yz) = (xy)z = xyz$ $\forall x, y, z \in A$

iii. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ $\forall x, y, z \in A$

iv. $\lambda(xy) = (\lambda x)y = x(\lambda y)\ \forall x, y \in A$ and $\lambda \in C$

Algebra with Hausdorff topology and continuous algebraic operations will be called topological algebra. It will be commutative if $xy = yx\ \forall x, y \in A$. A topological algebra will be called locally convex algebra if its topology is generated by the semi norms $\{p_\alpha\}_{\alpha \in I}$ such that $\forall \alpha \in I$

i. $p_\alpha(x) \geq 0$ and $p_\alpha(x) = 0$ if $x = 0$

ii. $p_\alpha(x + y) \leq p_\alpha(x) + p_\alpha(y)\ \forall x, y \in A$

iii. $p_\alpha(\lambda x) = |\lambda| p_\alpha(x)\ \forall x \in A$ and $\lambda \in C$.

A locally convex algebra will be called locally multiplicatively convex (lmc-algebra) if $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)\ \alpha \in I$ and $\forall x, y \in A$. A complete metrizable lmc-algebra a is called Frechet algebra. A * algebra or involution algebra is a complex algebra with an involution $*$ that is a mapping satisfying

i. $(x + y)^* = x^* + y^*$

ii. $(xy)^* = y^* x^*$

iii. $(\alpha x)^* = \overline{\alpha} x^*$ where $\overline{\alpha}$ is complex conjugate of α

iv. $(x^*)^* = x^{**} = x\ \forall x, y \in A$ and $\alpha \in C$

lmc*-algebra A is an lmc-algebra with an involution $*$ such that $p_\alpha(x*) = p_\alpha(x)\ x \in A, \alpha \in I$ that is involution $*$ is continuous. Δ will be the set of all non-zero continuous multiplicative linear functionals on a unital lmc*-algebra A. If A is a commutative locally multiplicatively convex algebra then; $rad(A) = \{x \in A : f(x) = 0 \forall f \in \Delta A\}$. In the case of $rad(A) = \{0\}$, it will be called **semi simple**. If A is a commutative locally multiplicatively convex algebra then to each $x \in A$, we define Gelfand transform of x as $\hat{x} : \Delta(A) \to C$ given by $\hat{x}(f) = f(x) \forall f \in \Delta A$. The set of all Gelfand transforms of A is denoted by $\hat{A}$ where $\hat{A} = \{\hat{x} : x \in A\}$. A function $f(x; y)$ is separately continuous if at any point the restricted functions $fx(y)$ and $fy(x)$ are continuous as functions of one variable, Zbigniew and Robert (2003). Regarding the general theory and in depth study of locally multiplicatively convex algebras, the reader is referred to Michael (1952), Anastasios and Marina (2007), Mallios (2003) and of Mathematics (2012)

3. Results and Discussion

Theorem 3.1. *Let A be a unital commutative separable Frechet Q lmc*- algebra. Then there exist a probability measure μ on Δ which is positive on non-empty open subsets of Δ.*

Proof. Δ is weak*-compact, Michael (1952) so it is metrizable, Rudin (1981), hence separable. Let $\{f_n\}$ be a dense subset of Δ. For any open subset $\emptyset \neq U \subset \Delta$ and each positive integer n we define;

$$\mu_n(U) = \begin{cases} 1 & f_n \in U \\ 0 & \text{otherwise} \end{cases}$$

then $\mu(U) = \sum\limits_{n=1}^{\infty} 3^{-n}\mu_n(U)$ which satisfy as required. QED. □

Theorem 3.2. *Let A be a unital commutative semi simple separable Frechet Q lmc*-algebra. For $x, y \in A, < x, y >= \int\limits_{\Delta} \hat{x}\overline{\hat{y}}d\mu$ where $\hat{x}, \hat{y}$ are Gelfand transformations of x, y respectively and μ is probability measure on Δ, is an inner product on A*

Proof. Let $< x, x >= 0$ then $\int\limits_{\Delta} |\hat{x}|^2 d\mu = 0$ then $\hat{x} = 0$ almost everywhere that is $\hat{x}(f) = 0$ for almost every $f \in \Delta$. We claim that $\hat{x}(f) = 0$ for every $f \in \Delta$ that is $x \cap \{ker(f) : f \in \Delta\} = R(A)$. Semi-simple do the job so $x = 0$ Now let $x \neq 0$ then $\exists f \in \Delta \ni \hat{x}(f) \neq 0$, then $\{f \in \Delta : \hat{x}(f) \neq 0\} \neq \Phi$. Since the mapping $f \to f(x) = \hat{x}(f)$ is continuous on Δ relative to weak*-topology so $\{f \in \Delta : \hat{x}(f) \neq 0\}$ is an open subset of Δ. By theorem 3.1, μ is positive this implies $\int\limits_{\Delta} |\hat{x}|^2 d\mu \neq 0$, a contradiction. QED. □

Theorem 3.3. *The inner product space $A_\mu = (A, \| * \|_\mu)$ where $\| * \|_\mu)$ is the norm generated by inner product in theorem 3.2, is a topological algebra.*

Proof. It will be sufficient to show that multiplication is separately continuous. Let x be a fixed point in A then for all $y \in A$ we have; $\|xy\|_\mu^2 = \int\limits_{\Delta} (\hat{xy})\overline{(\hat{xy})}d\mu = \int\limits_{\Delta} \hat{x}\hat{y}\overline{\hat{x}\hat{y}}d\mu = \int\limits_{\Delta} |\hat{x}|^2|\hat{y}|^2 d\mu \leq r_A(x)^2 \int\limits_{\Delta} \hat{y}|^2 d\mu = r_A(x)^2\|y\|_\mu^2$ where as $r_A(x)$ is the spectral radius of x and $|\hat{x}| \leq r_A(x)$. Hence, $\|xy\|_\mu^2 \leq r_A(x)^2\|y\|_\mu^2 \Rightarrow \|xy\|_\mu \leq r_A(x)\|y\|_\mu \Rightarrow \|xy\|_\mu \leq \|x\|_\mu\|y\|_\mu$. QED □

Theorem 3.4. *If A be a unital commutative symmetric Frechet Q lmc*- algebra then for all $x, y \in A$;*

i. $< x, y >=< y^*, x^* >$ *ii.* $\|x\|_\mu = \|x^*\|_\mu$

Proof. i. $< y^*, x^* >= \int\limits_{\Delta} \hat{y^*}\overline{\hat{x^*}} d\mu = \int\limits_{\Delta} \overline{\hat{y}\overline{\hat{x}}} d\mu = \int\limits_{\Delta} \hat{x}\overline{\hat{y}} d\mu =< x, y >$

ii. $\|x^*\|_\mu^2 =< x^*, x^* >= \int\limits_{\Delta} \hat{x^*}\overline{\hat{x^*}} d\mu = \int\limits_{\Delta} \overline{\hat{x}}\hat{x} d\mu = \int\limits_{\Delta} |\hat{x}|^2 d\mu =< x, x >= \|x\|_\mu^2$

QED. □

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